Methods of dynamical systems theory in modelling economic growth

Adam Krawiec

Institute of Public Affairs, Jagiellonian University

M. Kac Complex Systems Research Centre, Jagiellonian University

FENS, Kraków, 22 April 2006

Qualitative analysis of dynamical systems

The two-dimensional dynamical system is given by

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y).$$

Critical points are defined as $P(x, y) = Q(x, y) = 0$.
The character of a critical point is determined by eigenvalues of the linearization matrix of the system

$$B = \begin{bmatrix} \frac{\partial P}{\partial x}(x_0, y_0) & \frac{\partial P}{\partial y}(x_0, y_0) \\ \frac{\partial Q}{\partial x}(x_0, y_0) & \frac{\partial Q}{\partial y}(x_0, y_0) \end{bmatrix}$$

Eigenvalues λ are solutions of the characteristic equation

$$\lambda^2 - (\operatorname{tr} B)\lambda + \det B = 0$$

Qualitative analysis of dynamical systems

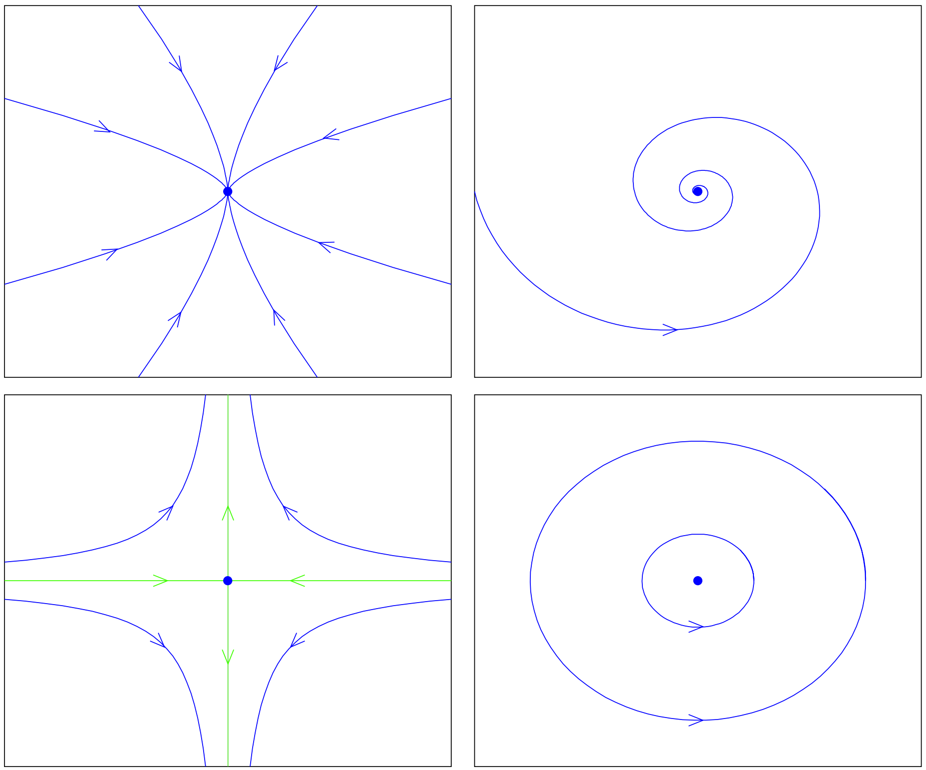
Eigenvalues

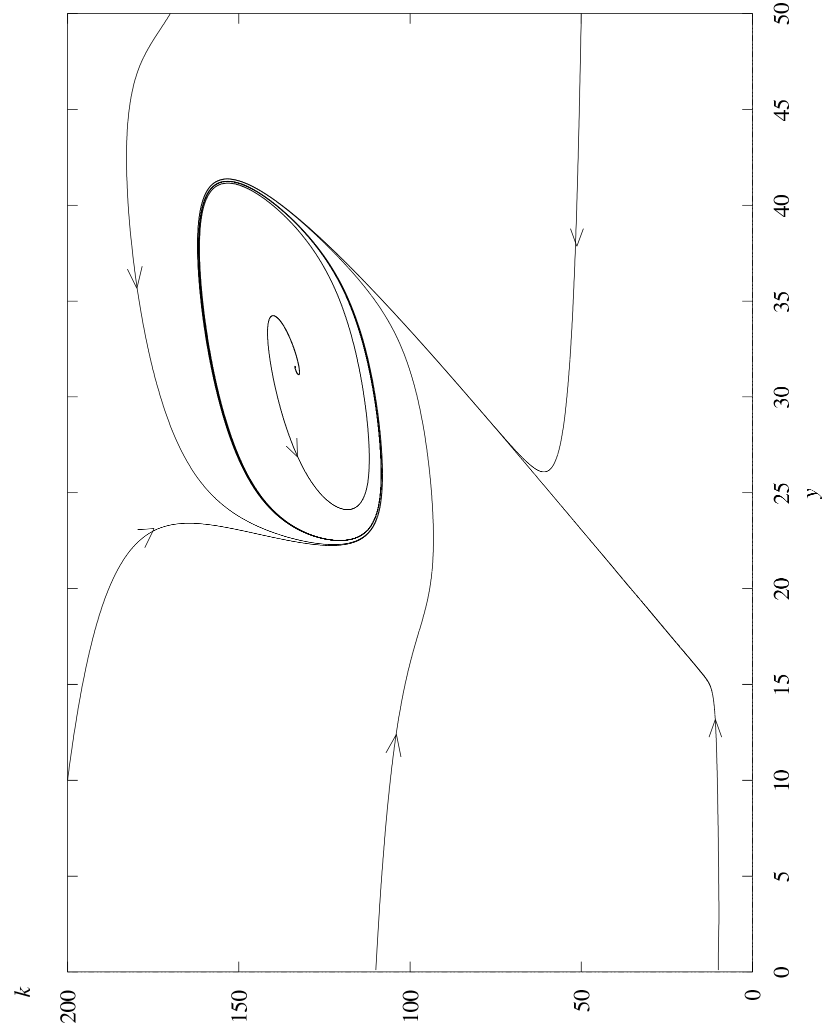
$$\lambda_{1,2} = \frac{\operatorname{tr} B \pm \sqrt{(\operatorname{tr} B)^2 - 4\det B}}{2}.$$

- For real eigenvalues (the discriminant is positive or zero)
  a node (eigenvalues of the same sign $\det B > 0$)
  a saddle (eigenvalues of different signs $\det B < 0$).
- For complex eigenvalues (the discriminant is negative)
  a focus ($\operatorname{Re}\lambda \neq 0$) or a centre ($\operatorname{Re}\lambda = 0$).

Depending on the sign of trace of linearization matrix nodes and focuses can be stable ($\operatorname{tr} B < 0$) or unstable ($\operatorname{tr} B > 0$).

Punkty krytyczne i cykl graniczny





Production function

- The production function depends on physical capital K, labour L, and knowledge A

$$Y(t) = F(K(t), A(t)L(t)).$$

- Capital are produced according to the Cobb-Douglass production function

$$Y(t) = K^{\alpha}(AL)^{1-\alpha}, \quad 0 < \alpha < 1.$$

- If we use quantities per unit of effective labour AL, then we have got

$$y(t) = k^{\alpha}.$$

Solow model of economic growth

- The equation for capital accumulation is given by

$$\dot{K} = sY(t) - \delta K(t)$$

- Labour and knowledge change as follows

$$\frac{\dot{L}}{L} = n, \qquad \frac{\dot{A}}{A} = g.$$

Finally, the Solow model is given as

$$\dot{k} = sk^{\alpha} - (n + g + \delta)k.$$

Model of growth with human capital

- The production function depends on physical capital K, human capital H, labour L, and knowledge A

$$Y(t) = F(K(t), H(t), A(t)L(t)).$$

- Labour and knowledge change as follows

$$\frac{\dot{L}}{L} = n, \qquad \frac{\dot{A}}{A} = g + \mu\frac{\dot{K}}{K} + \nu\frac{\dot{H}}{H}$$

- Both kinds of capital are produced according to the Cobb-Douglass production function (in effective labour AL unit)

$$y(t) = k^\alpha h^\beta, \quad 0 < \alpha + \beta < 1.$$

Capital dependent knowledge

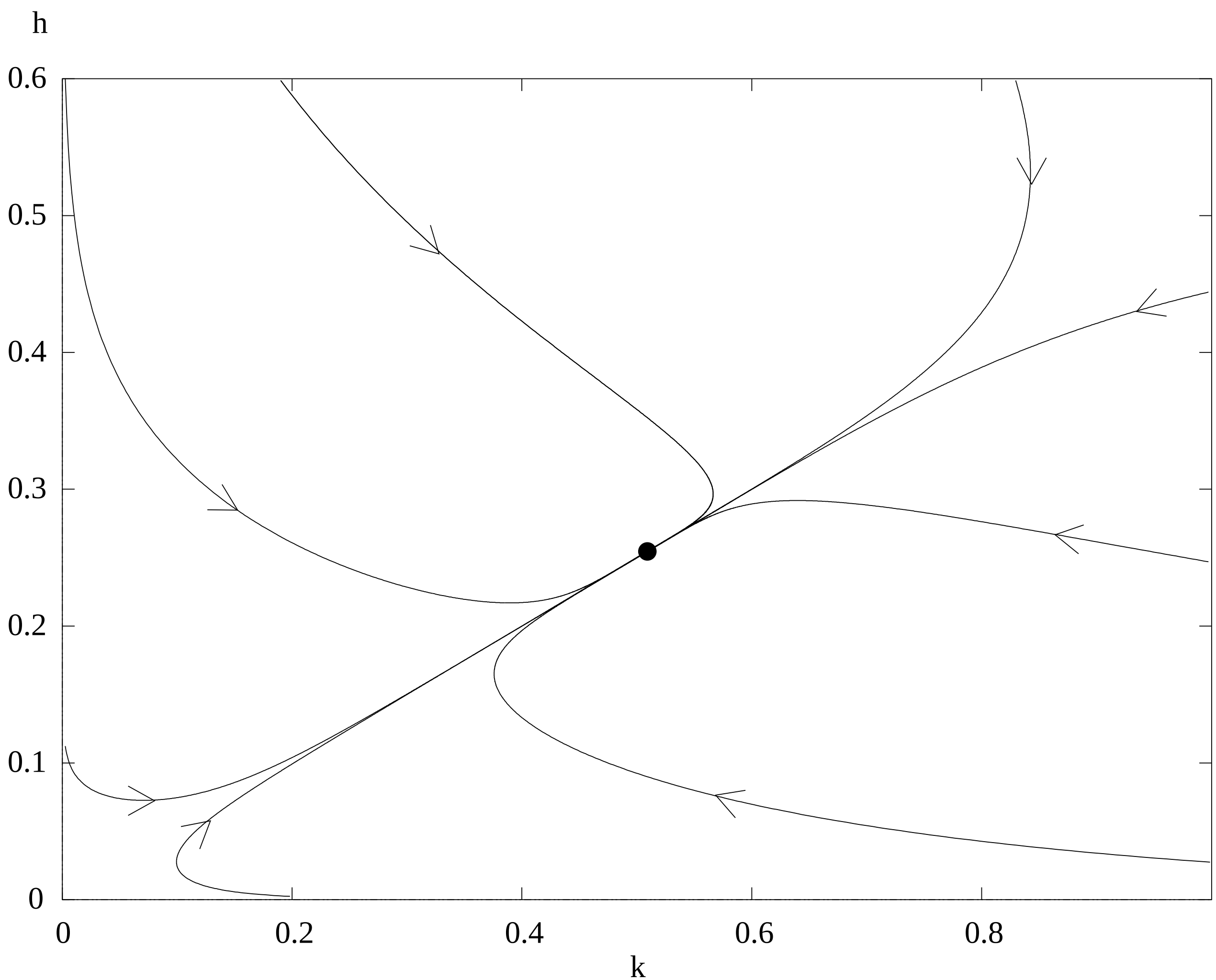
The model has the form of a dynamical system

$$\dot{k} = (1-\mu)s_k k^\alpha h^\beta - \nu s_h k^{\alpha+1} h^{\beta-1} - [(1-\mu-\nu)\delta + n + g]k$$
$$\dot{h} = (1-\nu)s_h k^\alpha h^\beta - \mu s_k k^{\alpha-1} h^{\beta+1} - [(1-\mu-\nu)\delta + n + g]h$$

There are at least two critical points in finite domain of phase space:

- the saddle (located at the origin: $k = 0$, $h = 0$)
- the stable node (for different values of the parameters the node is located on the the line $k \propto h$).

The phase portrait



Model with Exogenous Knowledge

Let knowledge grows at constant rate

$$\frac{\dot{A}}{A} = g.$$

The capital accumulation is given

$$\dot{k} = f(k(t)) - c - (g + n + \delta)k(t).$$

Households choose such a level of consumption over time to maximise their utility function

$$U = \int_0^\infty e^{-\rho t} u(C(t)) dt$$

where ρ is a discount rate.

Model with Exogenous Knowledge

We assume

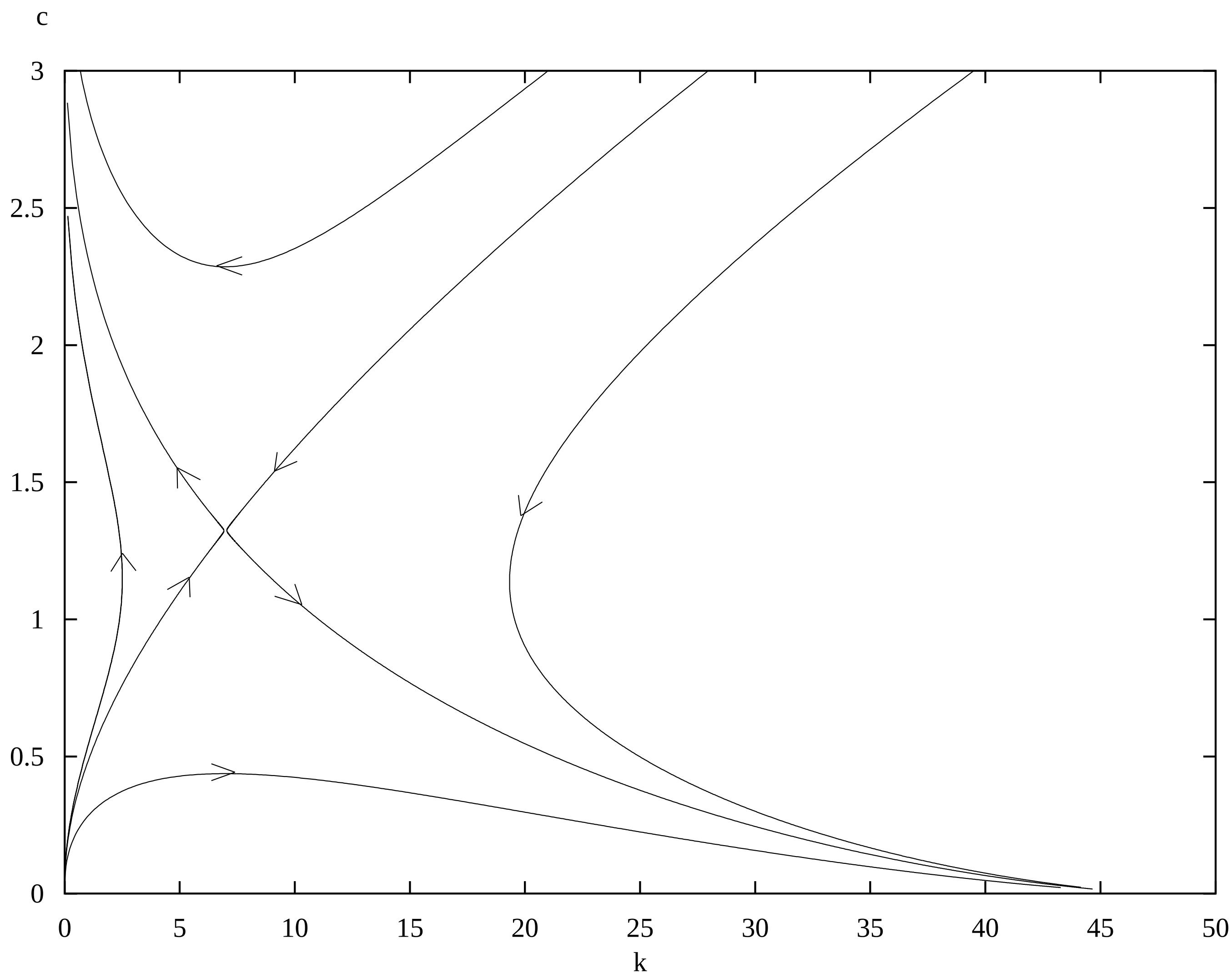
- the Cobb-Douglas production function $f(k) = k^{\alpha}$
- and the CRRA utility function $u(c(t)) = \frac{c(t)^{1-\sigma}}{1-\sigma}$.

We solve the maximisation problem and obtain

$$\begin{aligned} \dot{k} &= k^{\alpha} - c - (\delta + g + n)k \\ \dot{c} &= \frac{c}{\sigma}(\alpha k^{\alpha-1} - \delta - g - n - \rho). \end{aligned}$$

There are three critical points:
the unstable node, the stable node, and the saddle.

The phase portrait 2a



Endogenous technological progress

Let the rate of growth of physical capital has influence on knowledge

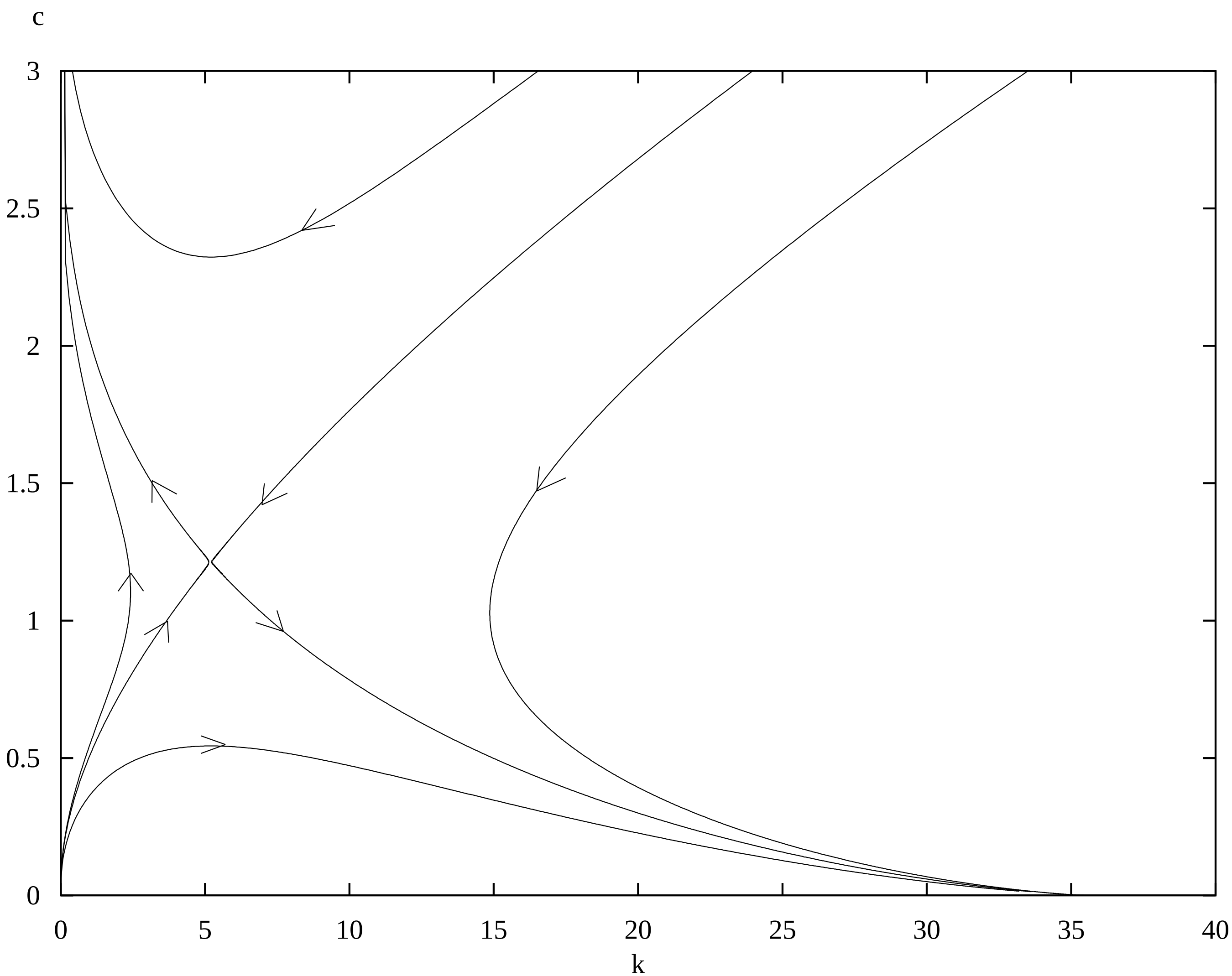
$$\frac{\dot{A}}{A} = g + \mu \frac{\dot{K}}{K}.$$

The optimisation procedure gives the two-dimensional dynamical system

$$\dot{k} = (1-\mu)k^{\alpha} - (1-\mu)c - [(1-\mu)\delta + g + n]k$$
$$\dot{c} = \frac{c}{\sigma}[\alpha(1-\mu)k^{\alpha-1} - (1-\mu)\delta - g - n - \rho].$$

There are three critical points:
the unstable node, the stable node, and the saddle.

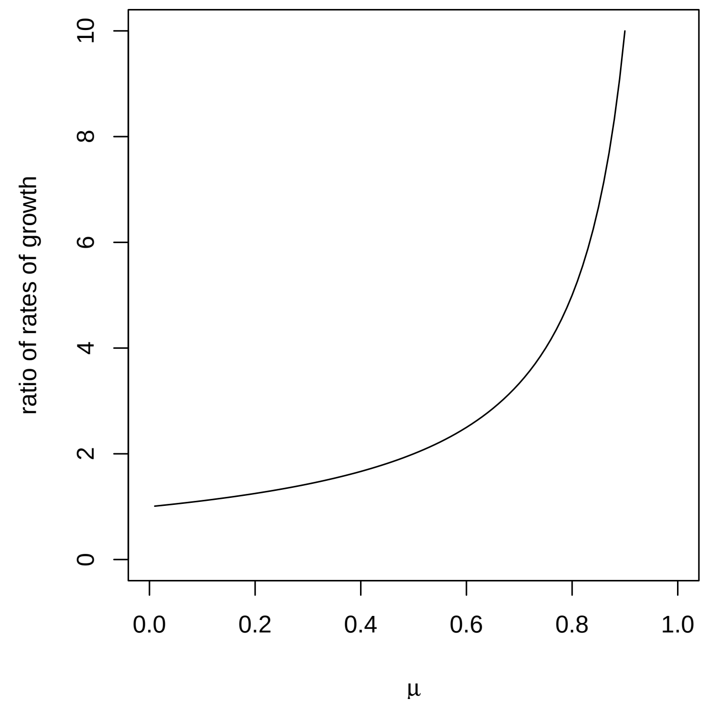
Phase portrait 2b



Ratio of rates of growth of K, C, Y

$$R_X = \frac{\frac{g+n}{1-\mu}}{g+n} = \frac{1}{1-\mu}$$

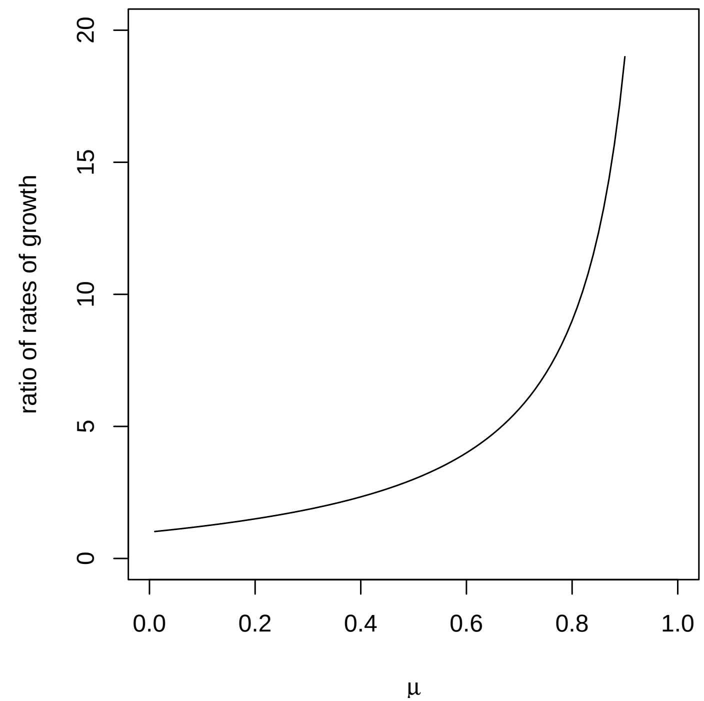
The ratio of rates of growth of capital K, consumption C, output Y in these two models depends only on the parameter μ.
For $\mu = 0.2$ the rate of growth is 25% higher,
for $\mu = 0.5$ the rate of growth is 2 times higher.



Ratio of rates of growth of k, c, y

$$R_{X/L} = \frac{\frac{g+\mu n}{1-\mu}}{g} = \frac{g+\mu n}{g(1-\mu)}.$$

The ratio of rates of growth of per capita capital k, consumption c, output y depends on μ, n and g.
When $g = n$
$\mu = 1/3$ the rate of growth is 2 times higher,
$\mu = 2/3$ the rate of growth is 5 times higher.



Conclusions

- We assume that the rate of change of knowledge can depend on both the rate of change of physical and human capital. The dynamics of these models was investigated in terms of dynamical system theory.
- We considered the simple model with knowledge dependent on the rate of change of physical and human capital, and studied the qualitative dynamics of the model on the phase portrait.
- We compared the Ramsey model of optimal economic growth and the model with knowledge dependent on physical capital. We found higher values of rates of change of the phase variables at the stationary equilibrium in the latter model.