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Mathematical Methods of Physics

Problems 5

5.1 Use the Fourier transform to show that the Maxwell's equations written in terms of the wave vector $\mathbf{k}$ have the form:

$$\mathbf{k} \cdot \mathcal{E}(\mathbf{k}, t) = 0\,,$$

$$i[\mathbf{k} \times \mathcal{E}(\mathbf{k}, t)] = -\frac{\partial}{\partial t}\mathcal{B}(\mathbf{k}, t)\,,$$

$$\mathbf{k} \cdot \mathcal{B}(\mathbf{k}, t) = 0\,,$$

$$ic^2[\mathbf{k} \times \mathcal{B}(\mathbf{k}, t)] = \frac{\partial}{\partial t}\mathcal{E}(\mathbf{k}, t)\,.$$

Assume that the Fourier transform of an electric field and a magnetic field at time t can be written as

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3k\, \mathcal{E}(\mathbf{k}, t)\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\,,$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3k\, \mathcal{B}(\mathbf{k}, t)\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\,.$$

5.2 Show that if $\psi(x)$ has the Fourier transform $\phi(k)$, the Parseval's identity holds

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, |\psi(x)|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}k\, |\phi(k)|^2\,.$$

5.3 Solve the non-homogeneous Helmholtz equation in the form

$$[\nabla^2 - \mu^2]\psi(\mathbf{r}) = S(\mathbf{r}) \,,$$

where $\mu \in \mathbb{R} \setminus \{0\}$, using the Fourier transform.

Assume that

$$\psi(\mathbf{r}) = \frac{1}{(2\pi)^3} \int \phi(\mathbf{k}) \, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}^3 k \,,$$

$$S(\mathbf{r}) = \frac{1}{(2\pi)^3} \int \mathcal{S}(\mathbf{k}) \, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}^3 k \,,$$

and take $S(\mathbf{r}) = \delta(\mathbf{r})$.

Hint:

$$\int \frac{k^2 \, \mathrm{d}k}{k^2 + \mu^2} = k - \mu \arctan\left(\frac{k}{\mu}\right).$$

Bartłomiej Spisak, Maciej Wołoszyn